Modeling and Analysis of Congestion Phenomena

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1. Abstract

Traffic jam, delivery delay, long line of people or ants, etc. Every one of them can be considered as a kind of congestion phenomena. They always cause the economic loss in the society. In this paper, we investigate the factor causing congestion. However, completely observing the phenomena directly in the real world is very difficult. Hence, we give mathematical models of the phenomena and analyze them. We define three models, each of which is characterized by their own sets of rules and devised the methods of analyzing the models. Experimental results indicate that certain changes of the rules can result in a significant difference in behavior of the model. And we found the same change also in the reality

2. Introduction

There are many various congestion phenomena in the world including Traffic jam, delivery delay, long line of people or ants, etc. We investigate the consistent property of them. However, they are complicated because they contain various aspects. It is very difficult to analyze them in the reality precisely. Therefore, we give some mathematical models (See Figure 1) and try to improve them.

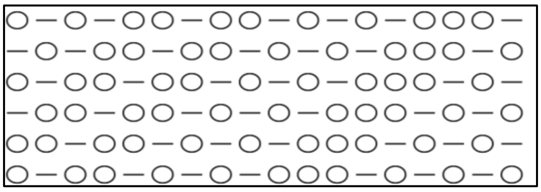

Figure1: Our model of congestions

The purpose of our study is to investigate properties and theories in some different models which contain the fundamental and essential nature.

3. Models

3.1 The definition of standard model

We model congestion phenomena using cellular automata, which is one of fundamental models in computer science. For convenience of explanation, we explain our model using traffic jam as an example of congestion phenomena. Each circle shows a car or in the traffic jam. This is called *agent*. Each stick shows the road. This is called *a place*. The circle decides whether to proceed or not by considering the distance between it and the one ahead.

3.2 The rules of movement.

・Each agent proceed by one step for each unit time.

・Each agent stays in the current position if the next position is occupies with an agent.

・The first space and the last space connect circularly.

3.3 The definition of the experiment model

We analyzed three different models which are changed the settings.

1. Basic model.

We first define the following two parameters.

C: The distance between two agents it can proceed.

P: The probability of proceeding under the conditions agent decided to proceed.

This is our basic model (See Figure 2).

(C = 1、P = 0.5)

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Figure 2: Our basic model

2. P linear model

The conditions agent can proceed reflect the function of the distance between two agents (See Figure 3).

The function: p(d) = Ad A:constant

(p(d) = 0.5d)

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Figure 3: Our linear model

3. Dull model

The agent cannot proceed if it stopped in previous step (See Figure 4).

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Figure4: Our dull model

4. Experiment

4.1 The methods

We carry out the programs of the three models. The programing language we used is Ruby 2.4.4. We define the following two functions to represent physical quantities in order to analyze the results.

$$\begin{cases} \text{Density } (x) = a / n, \\ \text{Flow rate } (y) = a_f / n, \end{cases}$$

where

n : The number of all the squares

a : The number of all the agents

a_f :The number of the agents which proceed by one step.

We gathered measured data of f-x changing the settings of the models. The initial placement of agents is not unique because even if the density is decided, we need to select which to arrange them evenly or to arrange them unevenly. Therefore we also gather measured data about the one which the initial placement is different. We call the farmer one *equal road* and the latter one *random road.*

4.2 The results of the experiment

・n=1000 The number of step is 10000.

・x-axis is density and y-axis is flow late.

・We make the data set which change c , p , or A.

1.The results of Basic model

The initial placement does not influence the results (See Figure5,6).

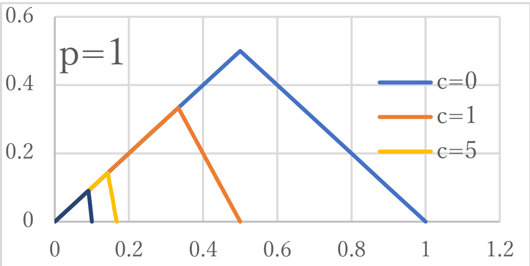

Figure:5 The result of basic model. (p = 1)

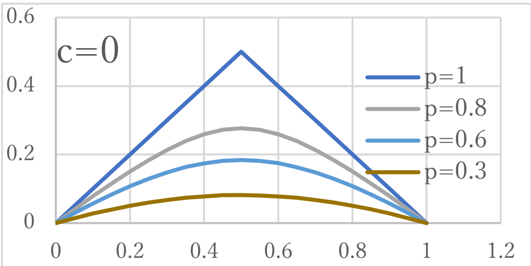

Figure:6 The result of basic model. (c = 0)

2.The results of P linear model

The initial placement does not influence the results (See Figure7)

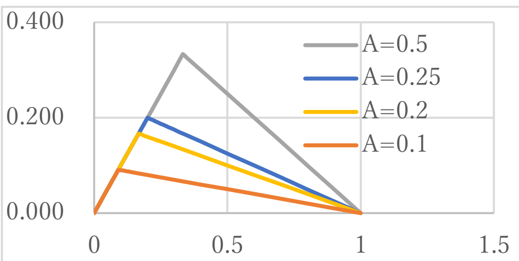

Figure:7 The result of linear model

3.The results of dull model

We get the two results which depend on the initial placement. We found only equal road keep flow rate high (See Figure8).

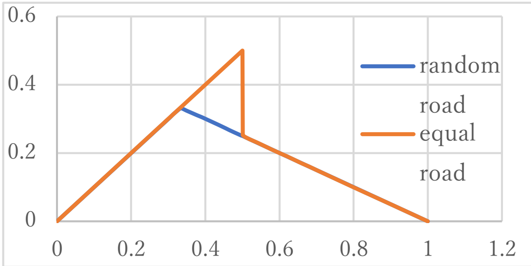

Figure:8 The result of dull model

5. The verifications

We consider making models as Markov chain in order to change the function into the equation. This contains two significance. The first one is the verification of the accuracy of the computer simulation. This is because we cannot affirm the accuracy of the graph because we carried out the experiment which contains stochastic process. The other is causing in the formula allows to compare maximum value or slope. This process is essential to consider the results.

5.1 The methods of the verification

Markov process is stochastic process as future states are depended on not past sates, but present state and, especially the one the states are discrete is called Markov chain. We aimed to show three models are Markov chain and by doing so we got some equations of the definitions and transitions. We solved simultaneous equations and showed flow rate as a function of x. We conducted this against all the three models. We show the concrete certification and the derivation process about Basic model and show only the results about the other ones.

5.2 The results of the verifications

1. Basic model

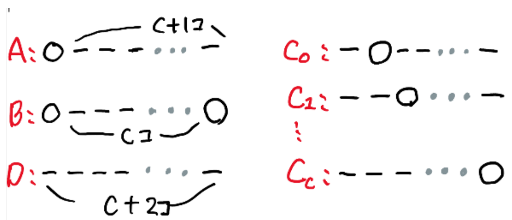

Figure:9 The states Xn

We classify the states X_n which are taken in time n as shown in the figure 9 above. We focus on the lead square and the classification is mutually exclusive. Considering the relations between the states X_n and X_{n+1}, the processes which can be existed are as shown in the figure 10 below.

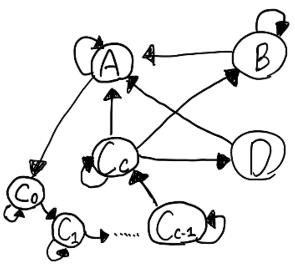

Figure:10 The processes

The probabilities of the processes are calculated as conditional ones. This is based on the definitions of the rules of Basic model. We define P; transition probability matrix. Than this We affirm the next states are depended on only the present ones. Therefore, this is Markov chain.

Considering the existence of stationary distribution, we represent conditional probability in the formula.

We show stationary distribution as

$$\pi = \begin{pmatrix} Pa \\ Pb \\ Pc_0 \\ \vdots \\ Pcc \\ Pd \end{pmatrix}.$$

We define matrix which ij component contains probability of the process from I state to j state as P. P is as shown in the figure in next page.

We get some equations about the definitions and certifications.

- $\pi P = \pi$
- $Pa + Pb + Pc_1 + \cdots + Pd = 1$
- $Pa + Pb = x$
- $Pc_k = x \quad (k = 0,1,\cdots,c-1)$
- $Pc_c = Pa$
- $f = p * Pa$

We solve simultaneous equations and get

$$f = \frac{-\sqrt{(c^2+4cp+p)x^2-2(c+2p)x+1} - \mathrm{c}x + 1}{2}$$

2. P linear model

$$f = \frac{-\sqrt{(A^2+2A+1)x^2-2A(A+1)x+A^2} - (A-1)x + A}{2}$$

3. Dull model

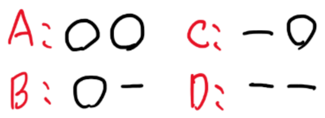

Figure:11 The states of agents

In this model, the next state cannot be depended on only the present state (See Figure11). Therefore this is not Markov chain and we affirm we cannot show the equation of flow rate in the same ways.

$$P = \begin{bmatrix} 1-p & 0 & p & \cdots & 0 & 0 \\ \dfrac{pPa}{Pa+Pb} & \dfrac{(1-p)Pa+Pb}{Pa+Pb} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \dfrac{(1-p)Pa+Pb}{Pa+Pb} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \dfrac{pPa}{Pa+Pd}\dfrac{pPa}{Pa+Pb} & \dfrac{pPa}{Pa+Pd} & 0 & \cdots & \dfrac{(1-p)Pa+Pb}{Pa+Pb} & \dfrac{pPa}{Pa+Pb} \\ \dfrac{pPa}{Pa+Pd} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

6. Considerations

6.1 evaluations

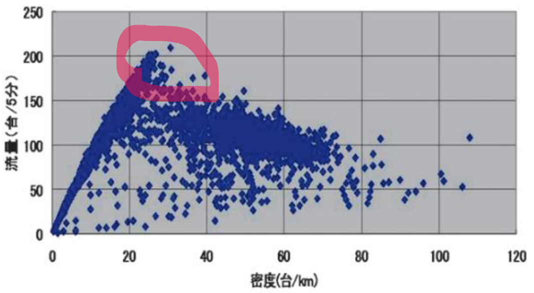

Figure:12 The traffic of the tome highway (Japan Highway Public Corporation)

The graph above is f-x graph which is measured in the reality. The line graph increases linearly at first, but when it reaches the limit it begins to decrease. The rate of change in decreasing is smaller than that in increasing. This reflects on A configuration of P linear model or Dull model.

The bump which appears in Dull model is shown as red circle as shown in figure above. The nature which the initial placement which is arranged evenly keep flow rate high even at a high density appears in the real traffic jam.

This is the example which shows traffic flow in the reality shows the same nature. Therefore, we cannot judge this nature is common to all congestion phenomena easily. However, this is a data in straight road which is insulated from the influence of the external environment and is considerable simplified data in real observation. Therefore, we consider this result can check usefulness of making models.

6.2 Theories

We show the theories which are gotten from analyzing the results under the trust which our models succeed in extracting the fundamental nature of the real congestion phenomena.

The rate of change in increase and decrease is inconsistent. The more C; critical distance, increase, the more the rate of change in decrease increase greatly. This is because if congestion phenomena happen even for just a bit, flow rate decrease rapidly, also decrease of flow rate of Dull model is larger than that of Basic model. Dull model is only seen difference of flow rate which is depended on difference of initial placement. Keeping even spacing makes flow rate considerable high state.

7.Conclusions

We gave the three mathematical models o of congestion phenomena and analyzed them mathematically. We proceeded with considerations and found some theories are established in various congestion phenomena, also we established basement we derive the function which cannot be expected by only carrying out programs from the theories.

The models we gave contain the fundamental nature which is extracted by very strong generalizations. Therefore, we need to make the concrete models which adapted each example in order to solve the problem which we cut the loss in traffic jam or signal transduction in the reality. However, our proposed methods in this research are useful and may make congestion phenomena in the reality suppress.

Acknowledgments

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Reference

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